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1. INTRODUCTION

This work defines a two-temperature, relativistic, and isotropic electron distribution function. This distribution characterizes a collisionless, isotropic, relativistic, and nonequilibrium electron gas consisting of two populations of electrons: one in temporal thermal equilibrium at a "cold" electron temperature, T , with spatial density, n , and the other at a "hot" electron temperature, T' , with spatial density, n' . Nonrelativistic two-temperature electron distributions have been previously used to characterize bremsstrahlung spectra radiated by laser-generated plasmas.¹⁻⁴

Recent Argus and Shiva laser-pellet plasma studies at Lawrence Livermore Laboratory have measured hot electron temperatures as high as 100 keV.⁵⁻⁶ At such an electron temperature, relativistic effects should begin to appear, since the energy equivalent is a significant fraction of the electron rest energy.

In the nonrelativistic calculations of Eidmann,² the electron distribution is Maxwellian at temperature T for electron energies less than some "knee energy," E_n ; continuous; and Maxwellian at some temperature $T' > T$ for energies greater than E_n . The knee energy, E_n , is simply a phenomenological parameter. More recently, Wickens, Allen, and Rumsby⁴ have used a distribution function in configuration space, which is explicitly the sum of two nonrelativistic Maxwellians—one at a cold temperature and the other at a hot temperature. This distribution function is characterized by the respective cold and hot electron temperatures and densities. The two-temperature electron distribution formulated in the present work addresses the distribution in momentum space, as does the work of Eidmann;² however, modifications are that the distribution be relativistic and characterized by the hot and cold electron spatial densities rather than by a phenomenological knee energy.

In a future report,^{7,*} calculations will be documented for bremsstrahlung spectra from relativistic two-electron temperature plasmas based on this distribution function. This work has been conducted in the spirit of Bekefi et al, that, "theoretical spectra must be generated for a variety of reasonable distribution functions of electron velocities."⁸

In section 2, the Jüttner distribution is reviewed.⁹⁻¹² This distribution is the relativistic generalization of the equilibrium Maxwell distribution. The associated relationship between temperature and average particle energy is also discussed.

In section 3, a two-temperature electron distribution is defined as the sum of two Jüttner

¹ George Bekefi, *Principles of Laser Plasmas*, Wiley-Interscience, New York (1976).

² K. Eidmann, *Plasma Physics*, 17 (1975), 121.

³ Kent Estabrook and W. L. Kruer, *Phys. Rev. Lett.*, 40 (1978), 42.

⁴ L. M. Wickens, J. E. Allen, and P. T. Rumsby, *Phys. Rev. Lett.* 41 (1978), 243.

⁵ B. L. Pruett, K. G. Tirsell, H. N. Kornblum, S. S. Glaros, and V. L. Slivinsky, *Bull. Am. Phys. Soc.*, 23, 7 (1978), 806.

⁶ H. N. Kornblum, B. L. Pruett, K. G. Tirsell, and V. W. Slivinsky, *Bull. Am. Phys. Soc.*, 23, 7 (1978), 806.

⁷ Howard E. Brandt, *Bull. Am. Phys. Soc.* 23, 7 (1978), 854.

⁸ George Bekefi, *Principles of Laser Plasmas*, Wiley-Interscience, New York (1976), 599.

⁹ George Bekefi, *Radiation Processes in Plasmas*, John Wiley, New York (1966), 49.

¹⁰ J. L. Synge, *The Relativistic Gas*, North-Holland, Amsterdam (1957).

¹¹ F. Jüttner, *Ann. d. Phys.* 34 (1911a), 856.

¹² W. Pauli, *Theory of Relativity*, Pergamon Press, New York (1958), 139.

* Part of this work was presented at the 20th Annual Meeting of the Division of Plasma Physics, Am. Phys. Soc. in Colorado Springs.

distributions: one at a cold temperature, T , and spatial density, n , and the other at a hot temperature, T' , and density, n' . The relationship between average electron energy and temperature is also determined. Finally, an expression for the "knee" kinetic energy (in the neighborhood of which the logarithm of the distribution function has a characteristic knee) is derived in terms of the cold and hot electron temperatures and densities.

2. THE JÜTTNER DISTRIBUTION

The electron distribution function for a relativistic electron gas in which the electrons are in their most random or equilibrium state is given by the Jüttner distribution,⁹⁻¹²

$$f(p, T, n) = n\beta[4\pi m^2 c K_2(\beta mc^2)]^{-1} \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} \quad (1)$$

Here, p is the magnitude of the relativistic electron momentum, n is the spatial electron density, T is the electron temperature, m is the mass of the electron, and c is the speed of light. The definition of β is

$$\beta = (kT)^{-1} \quad (2)$$

k is Boltzmann's constant, and $K_2(a)$ is the second-order modified Bessel function of the second kind defined by the definite integral¹³

$$K_2(a) = a\pi^{1/2}[2\Gamma(\frac{3}{2})]^{-1} \int_1^\infty dx x(x^2 - 1)^{1/2} \exp(-ax) \quad (3)$$

and

$$\Gamma(\frac{3}{2}) = \pi^{1/2}/2 \quad (4)$$

One observes that the relativistic distribution of equation (1) is of the canonical Maxwellian form,

$$f(p) = A \exp(-\beta E) \quad (5)$$

where A is a normalization constant and E is the single-particle energy given relativistically by

$$E(p) = [(mc^2)^2 + (pc)^2]^{1/2} \quad (6)$$

The normalization constant A is such that

$$\int d^3p f(p) = n \quad (7)$$

where the integral is over all of single-particle momentum space and n is the particle number density in space. Substituting equations (5) and (6) in the normalization condition, equation (7), then

$$A = n \left[4\pi \int_0^\infty p^2 dp \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} \right]^{-1} \quad (8)$$

⁹ George Bekefi, *Radiation Processes in Plasmas*, John Wiley, New York (1966), 49.

¹⁰ J. L. Synge, *The Relativistic Gas*, North-Holland, Amsterdam (1957).

¹¹ F. Jüttner, *Ann. d. Phys.*, **34** (1911a), 856.

¹² W. Pauli, *Theory of Relativity*, Pergamon Press, New York (1958), 139.

¹³ I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press (1965), 323.

Changing the variable of integration by

$$p = mc(x^2 - 1)^{1/2} , \quad (9)$$

and noting then that

$$dp = mcx(x^2 - 1)^{-1/2} dx , \quad (10)$$

then equation (8) becomes

$$A = n[4\pi(mc)^3 \int_1^\infty dx x(x^2 - 1)^{1/2} \exp(-\beta mc^2 x)]^{-1} . \quad (11)$$

Next, using the definition of the second-order modified Bessel function $K_2(a)$ of equation (3), then equation (11) becomes

$$A = n\beta[4\pi m^2 c K_2(\beta mc^2)]^{-1} . \quad (12)$$

Finally, combining equations (5), (6), and (12), one obtains the Jüttner distribution, equation (1).

The relationship between average electron kinetic energy and temperature for the Jüttner distribution differs significantly from that of the familiar nonrelativistic Maxwell distribution. For the relativistic gas, the average energy, \bar{E} , of an electron is given by

$$\bar{E} = n^{-1} \int d^3p E(p) f(p) , \quad (13)$$

where the single-particle energy $E(p)$ is given by equation (6), the integral is over all of single-particle momentum space, and $f(p)$ is the Jüttner distribution given by equation (1). Explicitly then, equation (13) becomes

$$\bar{E} = \int_0^\infty d^3p [(mc^2)^2 + (pc)^2]^{1/2} \beta [4\pi m^2 c K_2(\beta mc^2)]^{-1} \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} \quad (14)$$

One notes that equation (14) may be rewritten as

$$\bar{E} = -\beta[4\pi m^2 c K_2(\beta mc^2)]^{-1} (\partial/\partial\beta) \int_0^\infty d^3p \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} , \quad (15)$$

and substituting equation (12) in equation (15), then

$$\bar{E} = -An^{-1} (\partial/\partial\beta) \int_0^\infty d^3p \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} . \quad (16)$$

Next, substituting equation (8) in equation (16), then

$$\begin{aligned} \bar{E} = - \left[\int_0^\infty d^3p \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} \right]^{-1} (\partial/\partial\beta) \int_0^\infty d^3p \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} , \end{aligned} \quad (17)$$

or equivalently,

$$\bar{E} = -\partial/\partial\beta \ln Z , \quad (18)$$

where the quantity Z is defined by

$$Z = \int_0^\infty d^3p \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} . \quad (19)$$

Using equations (8) and (12) in equation (19), then

$$Z = \beta^{-1} [4\pi m^2 c K_2(\beta m c^2)] \quad (20)$$

Substituting equation (20) into equation (18), then

$$\bar{E} = -\partial/\partial\beta \ln\{\beta^{-1} [4\pi m^2 c K_2(\beta m c^2)]\} \quad (21)$$

or, simplifying equation (21), then

$$\bar{E} = \beta^{-1} + m c^2 \kappa(\beta m c^2) \quad (22)$$

where the function $\kappa(x)$ is defined by

$$\kappa(x) = -K'_2(x)/K_2(x) \quad (23)$$

According to the recursion relation for modified Bessel functions,¹⁴

$$K'_2(x) = -K_1(x) - 2x^{-1}K_2(x) \quad (24)$$

Substituting equation (24) in equation (23), then

$$\kappa(x) = 2x^{-1} + K_1(x)/K_2(x) \quad (25)$$

Finally, substituting equation (25) in equation (22), then

$$\bar{E} = 3\beta^{-1} + m c^2 K_1(\beta m c^2)/K_2(\beta m c^2) \quad (26)$$

Equation (26) relates the average electron energy to the temperature of the electron gas and the rest mass of the electron.

The electron kinetic energy, E_k , is of course given by the difference between its energy, E , and its rest energy, $m c^2$, namely,

$$E_k = E - m c^2 \quad (27)$$

Therefore, using equations (26) and (27), the average electron kinetic energy, \bar{E}_k , is given by

$$\bar{E}_k = 3\beta^{-1} + m c^2 [K_1(\beta m c^2)/K_2(\beta m c^2) - 1] \quad (28)$$

Clearly, in the nonrelativistic limit—namely, low temperature or equivalently $\beta m c^2 \gg 1$ —equation (28) must reduce to the classical equipartition relation, namely,

$$\bar{E}_k \xrightarrow{\beta m c^2 \gg 1} \left(\frac{3}{2}\right)\beta^{-1} \quad (29)$$

To see that this is indeed the case, one uses the following series expansion¹⁴ for $K_n(x)$.

$$\begin{aligned} K_n(x) = (\pi/2x)^{1/2} [\exp(-x)] \{ & 1 + (4n^2 - 1)[8x]^{-1} \\ & + (4n^2 - 1)(4n^2 - 9)[2!(8x)^2]^{-1} \\ & + (4n^2 - 1)(4n^2 - 9)(4n^2 - 25)[3!(8x)^3]^{-1} + \dots \} \end{aligned} \quad (30)$$

Using equation (30), it then follows that

$$K_1(x)/K_2(x) = 1 - \left(\frac{3}{2}\right)x^{-1} + \left(\frac{15}{8}\right)x^{-2} - \left(\frac{15}{8}\right)x^{-3} + O(x^{-4}) \quad (31)$$

¹⁴ M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55 (1964), 376-378.

Using equation (31) in equation (28) and combining terms, then for large βmc^2 one has that

$$\bar{E}_k = \left\{ \left(\frac{3}{2} \right) \beta^{-1} \right\} \left\{ 1 + \left(\frac{3}{2} \right) (\beta mc^2)^{-1} - \left(\frac{3}{2} \right) (\beta mc^2)^{-2} + O[(\beta mc^2)^{-3}] \right\} \quad (32)$$

Equation (29) then follows from equation (32).

3. THE RELATIVISTIC TWO-TEMPERATURE ELECTRON DISTRIBUTION

The Jüttner distribution, discussed in the preceding section, is the correct relativistic generalization of the nonrelativistic Maxwell-Boltzmann distribution for a relativistic gas.⁹⁻¹² The utility of two-temperature electron distributions in describing nonequilibrium laser plasmas was briefly discussed in section 1. In this section, a relativistic two-temperature electron distribution is defined and various properties are developed for the electron gas which it describes. In a future report, calculations will be documented for continuum x-ray spectra from relativistic two electron temperature plasmas, based on the distribution developed here.⁷

Motivated by nonrelativistic two electron temperature phenomenology referred to in section 1, one defines an isotropic relativistic two-temperature electron distribution function $f^{2T}(p)$ as the sum of two Jüttner distributions, $f(p, T, n)$ and $f(p, T', n')$, the one at a cold electron temperature, T , with spatial density, n , and the other at a hot electron temperature, T' , with spatial density n' ; thus,

$$f^{2T}(p) \equiv f(p, T, n, T', n') \equiv f(p, T, n) + f(p, T', n') \quad (33)$$

Appropriate normalization conditions are

$$\int d^3p f(p, T, n) = n \quad (34)$$

and

$$\int d^3p f(p, T', n') = n' \quad (35)$$

The total density N is

$$N = n + n' \quad (36)$$

Using equations (33) through (36), then

$$\int d^3p f^{2T}(p) = n + n' = N \quad (37)$$

Substituting equation (1) in equation (33), then

$$f^{2T}(p) = n\beta[4\pi m^2 c K_2(\beta mc^2)]^{-1} \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} \\ + n'\beta'[4\pi m^2 c K_2(\beta' mc^2)]^{-1} \exp\{-\beta'[(mc^2)^2 + (pc)^2]^{1/2}\} \quad (38)$$

where

$$\beta = (kT)^{-1} \quad (39)$$

and

$$\beta' = (kT')^{-1} \quad (40)$$

⁷ Howard E. Brandt, Bull. Am. Phys. Soc. 23, 7 (1978), 854.

⁹ George Bekefi, Radiation Processes in Plasmas, John Wiley, New York (1966), 49.

¹⁰ J. L. Synge, The Relativistic Gas, North-Holland, Amsterdam (1957).

¹¹ F. Jüttner, Ann. d. Phys., 34 (1911a), 856.

¹² W. Pauli, Theory of Relativity, Pergamon Press, New York (1958), 139.

Using the normalization equation (7) and equation (33), one easily sees that the normalization equation (37) is satisfied. Factoring the cold part of the distribution out of $f^{2T}(p)$ in equation (38), the latter can be rewritten in the following form

$$f^{2T}(p) = n\beta[4\pi m^2 c K_2(\beta mc^2)]^{-1} \exp\{-\beta[(mc^2)^2 + (pc)^2]^{1/2}\} \\ \cdot [1 + (n'/n)(\beta'/\beta)[K_2(\beta mc^2)/K_2(\beta' mc^2)] \\ \cdot \exp\{-(\beta' - \beta)[(mc^2)^2 + (pc)^2]^{1/2}\} \quad (41)$$

The two-temperature distribution $f^{2T}(p)$ describes the momentum space distribution of a relativistic gas of electrons consisting of two populations of electrons of spatial densities n and n' , respectively, both isotropically and Jüttner distributed, the first at a cold temperature, T , and the latter at a hot temperature, T' .

For the two temperature distribution, the average electron energy is

$$\bar{E} = N^{-1} \int d^3p [(mc^2)^2 + (pc)^2]^{1/2} f^{2T}(p) \quad (42)$$

Substituting equation (33) in equation (42), then

$$\bar{E} = N^{-1} \int d^3p [(mc^2)^2 + (pc)^2]^{1/2} [f(p, T, n) + f(p, T', n')] \quad (43)$$

Equation (43) can be rewritten in the following form,

$$\bar{E} = (n/N)\bar{E}(T) + (n'/N)\bar{E}(T') \quad (44)$$

where

$$\bar{E}(T) = n^{-1} \int d^3p [(mc^2)^2 + (pc)^2]^{1/2} f(p, T, n) \quad (45)$$

and

$$\bar{E}(T') = n'^{-1} \int d^3p [(mc^2)^2 + (pc)^2]^{1/2} f(p, T', n') \quad (46)$$

Here, $\bar{E}(T)$ and $\bar{E}(T')$ are the average energies of the cold and hot electrons, respectively. Using equations (45), (13), (6), and (26), then equation (45) becomes

$$\bar{E}(T) = 3\beta^{-1} + mc^2 K_1(\beta mc^2)/K_2(\beta mc^2) \quad (47)$$

Similarly, equation (46) becomes

$$\bar{E}(T') = 3\beta'^{-1} + mc^2 K_1(\beta' mc^2)/K_2(\beta' mc^2) \quad (48)$$

Substituting equations (47) and (48) in equation (44), and using equation (36), then one obtains the following expression for the average electron energy.

$$\bar{E}(T, n, T', n') = 3(n + n')^{-1}(n\beta^{-1} + n'\beta'^{-1}) \\ + mc^2(n + n')^{-1}[nK_1(\beta mc^2)/K_2(\beta mc^2) \\ + n'K_1(\beta' mc^2)/K_2(\beta' mc^2)] \quad (49)$$

The average kinetic energies are of course determined by equations (27), (47), (48), and (49). Thus, the average kinetic energy, $\bar{E}_k(T)$, of the cold electrons is given by

$$\bar{E}_k(T) = 3\beta^{-1} + mc^2[K_1(\beta mc^2)/K_2(\beta mc^2) - 1] \quad (50)$$

The average kinetic energy, $\bar{E}_k(T')$, of the hot electrons is given by

$$\bar{E}_k(T') = 3\beta'^{-1} + mc^2[K_1(\beta' mc^2)/K_2(\beta' mc^2) - 1] \quad (51)$$

The average electron kinetic energy, $E_k(T, n, T', n')$, is given by

$$E_k(T, n, T', n') = 3(n + n')^{-1}(n\beta^{-1} + n'\beta'^{-1}) - mc^2 \\ + mc^2(n + n')^{-1}[nK_1(\beta mc^2)/K_2(\beta mc^2) \\ + n'K_1(\beta' mc^2)/K_2(\beta' mc^2)] \quad (52)$$

The logarithm of the Jüttner equilibrium distribution $f(p)$, equation (4) or equation (5), as a function of energy (E or E_k) is linear, with slope $-\beta^{-1}$, just as is the case for the nonrelativistic Maxwell distribution. In the case of the relativistic two-temperature electron distribution, equation (41), for electron kinetic energies much less than some value E_{kk} , to be referred to here as the knee energy, the slope of $\ln f^{2T}$ as a function of E_k is nearly linear also, with approximate slope $-\beta^{-1}$. For electron kinetic energies $E_k \gg E_{kk}$, $\ln f^{2T}$ is also nearly linear, with slope $-\beta'^{-1}$. The knee energy, E_{kk} , is defined here as that electron kinetic energy at which the linear low and high-energy asymptotes intersect. Thus, by definition,

$$f^{2T} \xrightarrow[E_k \rightarrow E_{kk}]{} f_0^{2T}(E_k) = \ln f^{2T}(E_k = 0) + m_0 E_k \quad (53)$$

$$f^{2T} \xrightarrow[E_k \rightarrow E_{kk}]{} f_x^{2T}(E_k) = \ln f(E_k = 0, n', T') + m_x E_k \quad (54)$$

where

$$f_0^{2T}(E_{kk}) = f_x^{2T}(E_{kk}) \quad (55)$$

and

$$m_0 = \lim_{E_k \rightarrow 0} (\partial/\partial E_k) \ln f^{2T} \quad (56)$$

and

$$m_x = \lim_{E_k \rightarrow x} (\partial/\partial E_k) \ln f^{2T} \quad (57)$$

The electron "kinetic energy at the knee," E_{kk} , is defined by equation (55); namely, it is that kinetic energy at which the high energy asymptote, $f_x^{2T}(E_k)$ and the low energy asymptote, $f_0^{2T}(E_k)$, intersect. Clearly, there is considerable arbitrariness in defining the location of the knee. The definition employed here is both simple and adequate. Substituting equations (53) and (54) in equation (55), then

$$\ln f^{2T}(0) + m_0 E_{kk} = \ln f(0, n', T') + m_x E_{kk} \quad (58)$$

Solving equation (58) for the knee energy, E_{kk} , then

$$E_{kk} = (m_x - m_0)^{-1} \ln [f^{2T}(0)/f(0, n', T')] \quad (59)$$

Using equations (38), (6), and (27), one has that

$$f^{2T}(E_k) = n\beta[4\pi m^2 c K_2(\beta mc^2)]^{-1} \exp[-\beta(mc^2 + E_k)] \\ + n'\beta'[4\pi m^2 c K_2(\beta' mc^2)]^{-1} \exp[-\beta'(mc^2 + E_k)] \quad (60)$$

Evaluating equation (60) for $E_k = 0$, then

$$f^{2T}(0) = (4\pi m^2 c)^{-1} \{\beta n [K_2(\beta mc^2)]^{-1} \exp(-\beta mc^2) \\ + \beta' n' [K_2(\beta' mc^2)]^{-1} \exp(-\beta' mc^2)\} \quad (61)$$

Similarly, using equations (1), (6), and (27), then

$$f(E_k, n', T') = n' \beta' [4\pi m^2 c K_2(\beta' mc^2)]^{-1} \exp[-\beta'(mc^2 + E_k)] \quad (62)$$

and, therefore,

$$f(0, n', T') = n' \beta' [4\pi m^2 c K_2(\beta' mc^2)]^{-1} \exp(-\beta' mc^2) \quad (63)$$

Using equations (56) and (60),

$$m_0 = \lim_{E_k \rightarrow 0} (\partial/\partial E_k) \ln \{ \beta n [4\pi m^2 c K_2(\beta mc^2)]^{-1} \exp[-\beta(mc^2 + E_k)] \\ + \beta' n' [4\pi m^2 c K_2(\beta' mc^2)]^{-1} \exp[-\beta'(mc^2 + E_k)] \} \quad (64)$$

Reducing equation (64), it follows then that

$$m_0 = -\beta \{ 1 + (\beta'/\beta)(n'/n) [K_2(\beta mc^2)/K_2(\beta' mc^2)] \exp[(\beta - \beta')mc^2] \} \\ \cdot \{ 1 + (\beta'/\beta)(n'/n) [K_2(\beta mc^2)/K_2(\beta' mc^2)] \exp[(\beta - \beta')mc^2] \}^{-1} \quad (65)$$

The limiting form of $K_n(x)$ for small argument is given by¹⁴

$$K_n(x) \xrightarrow{x \rightarrow 0} [\Gamma(n/2)(x/2)^{-n}] \quad (66)$$

Using equations (65) and (66), one sees that for $\beta'/\beta \ll 1$, $n'/n \ll 1$, and $\beta mc^2 \ll 1$, then one has approximately

$$m_0 \approx -\beta \quad (67)$$

as stated above. Similarly, using equations (57) and (60), and realizing that $\beta' \approx \beta$, one obtains

$$m_0 \approx -\beta' \quad (68)$$

Finally, substituting equations (61), (63), (65), and (68) in equation (59), then the knee energy E_{kk} becomes

$$E_{kk} = \beta^{-1} (1 - \beta'/\beta)^{-1} \{ 1 + (\beta'/\beta)(n'/n) [K_2(\beta mc^2)/K_2(\beta' mc^2)] \exp[(\beta - \beta')mc^2] \} \\ \cdot \ln \{ 1 + (n/n')(\beta/\beta') [K_2(\beta' mc^2)/K_2(\beta mc^2)] \exp[-(\beta - \beta')mc^2] \} \quad (69)$$

Equation (69) expresses the knee kinetic energy in terms of the hot and cold electron parameters. This equation, together with equations (53), (54), (61), (63), (65), and (68), is useful in estimating the effects of hot electrons on the distribution function.

4. CONCLUSION

A relativistic two-temperature electron distribution, equation (41), has been motivated and defined. Expressions have been obtained for the average electron, cold electron, and hot electron energies, equations (49), (47), and (48), respectively. Corresponding average kinetic energies are given by equations (52), (50), and (51), respectively. These equations are useful in interpreting x-ray diagnostics of laser plasmas.⁷ Provided the electron distribution is of the assumed form, the inverse low and high energy slopes of the logarithm of the continuous x-ray spectrum measure the cold and hot electron temperatures, respectively, and equations (47) through (52) then deter-

¹⁴ M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55 (1964), 375.

⁷ Howard E. Brandt, *Bull. Am. Phys. Soc.*, 23, 7 (1978), 854.

mine the respective average electron energies. Also, the knee of the electron distribution function is determined by equation (69). Expressions (65), (67), and (68) were also obtained for the low and high energy asymptotic slopes of the logarithm of the electron distribution function. The logarithm of the distribution function as a function of electron kinetic energy can thus be characterized as approximately linear, with inverse slope equal to the cold temperature up to the knee energy and linear with inverse slope equal to the hot temperature for kinetic energies greater than the knee energy. In a future report, calculations will be documented of bremsstrahlung spectra from relativistic two electron temperature plasmas based on this electron distribution.

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